EXTENSIONS OF S5

By

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EXTENSIONS OF S5[1]

Dugundji has proved[2] that none of the Lewis systems of modal logic, S1 through S5,[3] has a finite characteristic matrix. The question arises whether there exist extensions of S5 which have no finite characteristic matrix. By an extension of a sentential calculus S, we usually refer to any system S' such that every formula provable in S is provable in S'. The answer to the question is trivially negative, in case we make no additional restrictions of the class of extensions. Thus the extension of S5 obtained by adding to the provable formulas the additional formula

$$p$$

has no finite characteristic matrix (indeed, it has no characteristic matrix at all), but this extension is not closed under substitution -- the formula

$$q$$

is not provable in it. McKinsey and Tarski have defined normal extensions of S4[4] by imposing three conditions. Normal extensions must be closed under substitution, must preserve the rule of detachment under material implication, and must also preserve the rule that if α is provable then $\sim\Diamond\sim\alpha$ is provable.[5] McKinsey and Tarski also gave an example of an extension of S4 which satisfies the first two of these conditions but not the third. One of the results of this paper is that every extension of S5 which satisfies the first two of these conditions also satisfies the third, and hence the above definition of normal extension is redundant for S5. We shall therefore limit the extensions discussed in this paper to those which are closed under substitution and which preserve the rule of detachment under material implication. These extensions we shall call quasi-normal. The class of quasi-normal extensions of S5 is a very broad class and actually includes all extensions which are likely to prove interesting. It is easily shown that quasi-normal extensions of S5 preserve the rules of replacement, adjunction, and detachment under strict implication.[6] It is the purpose of this paper to prove that every quasi-normal extension of S5 has a finite characteristic matrix and that every quasi-normal extension of S5 is a normal extension of S5 and to describe a simple class of characteristic matrices for S5.

Throughout the paper the terms formula, matrix, and satisfies will be employed in the usual way.[7] A matrix will be called an S-matrix if it satisfies all of the provable formulas of the system S. A matrix will be called a sub-matrix of a matrix $\mathfrak{M}$ if it is generated by elements of $\mathfrak{M}$ and contains as a designated class all elements of the generated matrix which are designated in $\mathfrak{M}$. Also we shall require two special classes of matrices, normal matrices and Henle matrices. The definition of a normal matrix is given by McKinsey;[8] that for a Henle matrix occurs later in the paper. The proof of Theorem 1 makes use of several references to papers of the Lewis systems and the matrices for these systems. Although many of of these papers do not concern S5 or its extensions, the proofs given and the constructions made require only properties which hold in S5 and all stronger systems.

Since it has been proved that the class of elements of a normal S2-matrix is a boolean algebra with respect to those operations of the matrix which correspond to $\sim$ and $\cdot$ in the logic,[9] we shall speak of the boolean algebra of the elements of a matrix and, when the matrix is finite, of the atoms of this algebra. It is convenient to distinguish between the operations of the logic and those of the matrices, and we shall use $\times$ as the matrix operation corresponding to $\cdot$ in the logic, $-$ for $\sim$, $*$ for $\Diamond$, and $\leftrightarrow$ for $\equiv$.

<u>Theorem 1</u>. If S is a quasi-normal extension of S5 and α is not provable in S, then α fails for a finite normal S-matrix.

<u>Proof</u>. By a construction due to Lindenbaum it is possible to show that if S is any quasi-normal extension of S5, then there exists an infinite characteristic matrix, $\mathfrak{M}$, for S.[10] By a construction due to McKinsey which modifies $\mathfrak{M}$, it can be shown that there exists an infinite, normal, characteristic matrix, $\mathfrak{M}'$, for S.[11] McKinsey and Tarski have proved[12] that every formula of S5 is equivalent to a formula of the first degree.[13] Now the number of formulas of less than the second degree containing n variables is finite and, therefore, the number of non-equivalent formulas containing n variables is finite. (The formulas α and β are said to be <u>equivalent</u> if $\alpha \equiv \beta$ is provable.) This result holds <u>a fortiori</u> for normal extensions of S5. Since in every normal S-matrix

$a \leftrightarrow b \in D$ if and only if $a = b$, it is established that every finite set of elements of a normal S-matrix generates a finite matrix which is clearly a normal S-matrix.

Now let α be a formula of S containing n variables which is not provable in S. There must be some substitution $(a_1, a_2, \ldots, a_n)$ of elements of $\mathcal{M}'$ for the variables of α such that $\alpha(a_1, a_2, \ldots, a_n) \notin D$. Let K be the sub-matrix of $\mathcal{M}'$ generated by $(a_1, a_2, \ldots, a_n)$. Then by the previous remarks, K is a finite normal S-matrix, and by construction α fails for this matrix.

Before stating Theorem 2 it is convenient to prove some Lemmas.

<u>Lemma 1</u>. If $\mathcal{M} = \langle K, D, X, -, * \rangle$ is a normal S5-matrix and a and b are elements of K such that $a \leq *b$, then $*a \leq *b$.

<u>Proof</u>. McKinsey has shown for S2-matrices that if $a \leq b$ then $*a \leq *b$.[14] Thus since $a \leq *b$ we have $*a \leq **b$. In S5 $\Diamond\Diamond p \equiv \Diamond p$ is provable and therefore $**x \leftrightarrow *x \in D$ and, since $\mathcal{M}$ is normal, $**x = *x$ for all $x \in K$. Thus we conclude that $*a \leq *b$.

<u>Lemma 2</u>. If a and b are atoms of K and $a \leq *b$, then $*b \leq *a$.

<u>Proof</u>. Since b is an atom, we know that either $b \leq *a$ or $b \leq -*a$. Suppose that $b \leq -*a$; then $b \times *a = 0$. Since $\mathcal{M}$ is normal and $\Diamond p \cdot \Diamond q \equiv \Diamond(p \cdot \Diamond q)$ is provable in S5, we have $*b \times *a = *(b \times *a) = *0 = 0$.[15] But by Lemma 1 since $a \leq *b$, we have $*a \leq *b$ and $*a \times *b = *a \neq 0$ which is a contradiction. Therefore $b \nleq -*a$ and we must have $b \leq *a$, and hence $*b \leq *a$, by Lemma 1.

<u>Lemma 3</u>. If a and b are atoms of K and $a \leq *b$, then $*a = *b$.

Proof. This follows immediately from Lemmas 1 and 2.

Lemma 4. If a and b are atoms of K, then either $*a = *b$ or $*a \times *b = 0$.

Proof. Since a is an atom, either $a \leq *b$ or $a \leq -*b$. If $a \leq *b$, then by Lemma 3 $*a = *b$. If, on the other hand, $a \leq -*b$, then $a \times *b = 0$; and, since $\mathcal{M}$ is normal, $*a \times *b = *(a \times *b) = *0 = 0$.

Lemma 5. If $\mathcal{M} = \langle K, D, \times, -, * \rangle$ is a finite, normal S5-matrix, then there exist elements $c_1, c_2, \ldots, c_r$ of K such that:

(i) $\sum_{i=1}^{r} c_i = 1$.

(ii) $c_i \times c_j = 0$ when $i \neq j$.

(iii) If $x \neq 0$ and $x \leq c_i$, then $*x = c_i$.

(iv) $c_i \neq 0$.

Proof. Denote by A the set of all atoms of K and let $\{c_1, c_2, \ldots, c_r\}$ be the set containing $*a$ for each $a \in A$. Since $a \leq *a$ for all $a \in A$ and $*a \in \{c_1, c_2, \ldots, c_r\}$ for all $a \in A$, we have $\sum_{a \in A} a \leq \sum_{i=1}^{r} c_i$. However, $\sum_{a \in A} a = 1$; and thus $\sum_{i=1}^{r} c_i = 1$, which is condition (i).

By Lemma 4, if a and b are atoms of K, then either $*a = *b$ or $*a \times *b = 0$. Thus if c_i and c_j are distinct members of $\{c_1, c_2, \ldots, c_r\}$, then $c_i \times c_j = 0$, which is condition (ii).

Suppose $x \leq c_i$. The element x is the sum of a finite number of atoms, $a_1, \ldots, a_k$, and for each atom, $a_j \leq c_i$. Since $\mathfrak{M}$ is normal we have $*x = *a_1 + \ldots + *a_k = c_i + \ldots + c_i = c_i$,[16] which is condition (iii).

Since c_i for $i = 1, 2, \ldots, r$ is equal to $*a$ for some atom a of K, condition (iv) is also satisfied.

Lemma 6. If $c_1, c_2, \ldots, c_r$ are elements of K satisfying Lemma 5 and $d = \prod_{x \in D} x$, then $c_i \times d \neq 0$ for $i = 1, 2, \ldots, r$.

Proof. Suppose that $c_k \times d = 0$, then $d \leq -c_k$. McKinsey has shown that if $d \leq x$ then $x \in D$,[17] and from this we get $-c_k \in D$. Let a be any atom of c_k, then $*a = c_k$ and thus $-*a \in D$. But McKinsey has also shown that $-*x \in D$ if and only if $x = 0$.[18] Therefore, $a = 0$ which is absurd since a is an atom.

We make three new definitions at this point. While the second two terms defined are common in mathematics, it seems desirable to define them here in order to make clear the manner in which the designated class is handled.

Definition 1. $\mathfrak{M} = \langle K, D, \times, -, * \rangle$ is a Henle matrix[19] if and only if:

(i) $\mathfrak{M}$ is a normal S5 matrix.

(ii) $*x = 1$ when $x \neq 0$.

Definition 2. If $\mathfrak{M}_1 = \langle K_1, D_1, \times_1, -_1, *_1 \rangle \ldots \mathfrak{M}_k$

$$= \langle K_k, D_k, \times_k, -_k, *_k \rangle$$

is a sequence of matrices, their direct product is the matrix

$\mathfrak{M} = \langle K, D, \times, -, * \rangle$ where:

(i) K is the set of all ordered k-tuples, $\langle x_1, x_2, \ldots, x_k \rangle$ such that $x_i \in K_i$ for $i = 1, 2, \ldots, k$.

(ii) D is the set of all ordered k-tuples $\langle y_1, y_2, \ldots, y_k \rangle$ such that $y_i \in D_i$ for $i = 1, 2, \ldots, k$.

(iii) $\langle x_1, x_2, \ldots, x_k \rangle \times \langle y_1, y_2, \ldots, y_k \rangle$

$$= \langle x_1 \times_1 y_1, x_2 \times_2 y_2, \ldots, x_k \times_k y_k \rangle.$$

(iv) $-\langle x_1, x_2, \ldots, x_k \rangle = \langle -_1 x_1, -_2 x_2, \ldots, -_k x_k \rangle$.

(v) $*\langle x_1, x_2, \ldots, x_k \rangle = \langle *_1 x_1, *_2 x_2, \ldots, *_k x_k \rangle$.

It is an immediate consequence of Definition 2 that a formula is satisfied by $\mathfrak{M}$ if and only if it is satisfied by $\mathfrak{M}_i$, for $i = 1, \ldots, k$.

<u>Definition 3</u>. The matrices $\mathfrak{M} = \langle K, D, \times, -, * \rangle$ and $\mathfrak{M}' = \langle K', D', \times', -', *' \rangle$ are <u>isomorphic</u> if and only if there exists a single-valued function F mapping K onto K' such that:

(i) $F(x \times y) = F(x) \times' F(y)$

(ii) $F(-x) = -'F(x)$

(iii) $F(*x) = *'F(x)$

(iv) $F(x) \in D'$ if and only if $x \in D$.

We are now in a position to prove

<u>Theorem 2</u>.[20] If $\mathfrak{M} = \langle K, D, \times, -, * \rangle$ is a finite normal S5-matrix, then it is isomorphic to the direct product of a finite sequence of Henle matrices.

<u>Proof</u>. Let $c_1, c_2, \ldots, c_r$ be the elements of K satisfying the conditions of Lemma 5. For $i = 1, 2, \ldots, r$ define the matrix

$\mathcal{M}_i = \langle K_i, D_i, \times_i, -_i, *_i \rangle$ where:

(i) $x \in K_i$ if and only if $x \leq c_i$

(ii) D_i = the set of all elements x of K_i such that $d_i \leq x$, where $d_i = d \times c_i$ and $d = \prod_{x \in D} x$.

(iii) $x \times_i y = x \times y$

(iv) $*_i x = *x$

(v) $-_i x = c_i \times -x$.

By Lemma 6, D_i is not vacuous for any i. It is easily shown that the matrices of this sequence are Henle matrices.

Let $\mathcal{M}' = \langle K', D', \times', -', *' \rangle$ be the direct product of these matrices. We wish to show that $\mathcal{M}$ is isomorphic to $\mathcal{M}'$.

For $x \in K$ let $F(x) = \langle c_1 \times x, c_2 \times x, \ldots, c_r \times x \rangle$. Obviously if $x \in K$, then $F(x) \in K'$. Suppose $\langle y_1, y_2, \ldots, y_r \rangle \in K'$, then by (i) above $y_i = c_i \times y_i$ for $i = 1, 2, \ldots, r$ and $c_j \times y_i = 0$ for $i \neq j$, and thus $\langle y_1, y_2, \ldots, y_r \rangle = F(y_1 + y_2 + \ldots + y_r)$ and $y_1 + y_2 + \ldots + y_r \in K$. Suppose $F(x) = F(y)$, then
$\langle c_1 \times x, c_2 \times x, \ldots, c_r \times x \rangle = \langle c_1 \times y, c_2 \times y, \ldots, c_r \times y \rangle$.
Thus $c_i \times x = c_i \times y$ for $i = 1, 2, \ldots, r$ and hence
$\sum_{i=1}^{r} c_i \times x = \sum_{i=1}^{r} c_i \times y$. Therefore we have

$$x = x \times 1 = x \times \sum_{i=1}^{r} c_i = \sum_{i=1}^{r} c_i \times x = \sum_{i=1}^{r} c_i \times y = y \times \sum_{i=1}^{r} c_i = y \times 1 = y.$$

This shows that the function F is a single valued function mapping K onto K'.

We now show that the equations of Definition 3 hold.

$$F(x \curlywedge y) = \langle c_1 \curlywedge (x \curlywedge y),\ c_2 \curlywedge (x \curlywedge y),\ \ldots,\ c_r \curlywedge (x \curlywedge y) \rangle$$

$$= \langle (c_1 \curlywedge x) \curlywedge (c_1 \curlywedge y), (c_2 \curlywedge x) \curlywedge (c_2 \curlywedge y), \ldots, (c_r \curlywedge x) \curlywedge (c_r \curlywedge y) \rangle$$

$$= \langle c_1 \curlywedge x, c_2 \curlywedge x, \ldots,\ c_r \curlywedge x \rangle \curlywedge' \langle c_1 \curlywedge y, c_2 \curlywedge y, \ldots, c_r \curlywedge y \rangle$$

$$= F(x) \curlywedge' F(y).$$

$$F(-x) = \langle c_1 \curlywedge -x,\ c_2 \curlywedge -x,\ \ldots,\ c_r \curlywedge -x \rangle$$

$$= \langle c_1 \curlywedge -(c_1 \curlywedge x), c_2 \curlywedge -(c_2 \curlywedge x),\ \ldots,\ c_r \curlywedge -(c_r \curlywedge x) \rangle$$

$$= \langle -_1(c_1 \curlywedge x),\ -_2(c_2 \curlywedge x),\ \ldots,\ -_r(c_r \curlywedge x) \rangle$$

$$= -' \langle c_1 \curlywedge x,\ c_2 \curlywedge x,\ \ldots,\ c_r \curlywedge x \rangle$$

$$= -'F(x).$$

$$F(*x) = \langle c_1 \curlywedge *x,\ c_2 \curlywedge *x,\ \ldots,\ c_r \curlywedge *x \rangle$$

$$= \langle *(c_1 \curlywedge x),\ *(c_2 \curlywedge x),\ \ldots\ ,\ *(c_r \curlywedge x) \rangle$$

$$= *' \langle c_1 \curlywedge x,\ c_2 \curlywedge x,\ \ldots\ ,\ c_r \curlywedge x \rangle$$

$$= *'F(x).$$

It remains to prove that $F(x) \in D'$ if and only if $x \in D$. We first note that McKinsey has proved[21] that for every finite S2-matrix a sufficient and necessary condition that $x \in D$ is that $d \leq x$ where $d = \prod_{y \in D} y$. *A fortiori* this is true for S5 matrices.

First, suppose $x \in D$. Then $d \leq x$ and $c_i \curlywedge d \leq c_i \curlywedge x$ for

$i = 1, 2, \ldots, r$, and thus, by (ii) above, $c_i \times x \in D_i$ for $i = 1, 2, \ldots, r$. Since $F(x) = \langle c_1 \times x, c_2 \times x, \ldots, c_r \times x \rangle$, $F(x) \in D'$.

Next, suppose $F(x) \in D'$. Then $c_i \times x \in D_i$ for $i = 1, 2, \ldots, r$ and $c_i \times d \leq c_i \times x$ for $i = 1, 2, \ldots, r$. Therefore $\sum_{i=1}^{r} c_i \times d \leq \sum_{i=1}^{r} c_i \times x$. But $\sum_{i=1}^{r} c_i \times d = d \times \sum_{i=1}^{r} c_i = d \times 1 = d$, and similarly $\sum_{i=1}^{r} c_i \times x = x$. Therefore $d \leq x$ and $x \in D$. This completes the proof of the theorem.

Still another theorem is required before the final results can be established. Several lemmas precede the proof of this theorem.

Lemma 7. Let $\mathcal{M} = \langle K, D, \times, -, * \rangle$ be a finite Henle matrix, a and b atoms of K, F_{ab} a function defined over the elements of K by the equations:

(i) $F_{ab}(x) = x$ if $a \leq x$ and $b \leq x$.

(ii) $F_{ab}(x) = x$ if $a \not\leq x$ and $b \not\leq x$.

(iii) $F_{ab}(x) = (x \times -a) + b$ if $a \leq x$ and $b \not\leq x$.

(iv) $F_{ab}(x) = (x \times -b) + a$ if $a \not\leq x$ and $b \leq x$.

Then, if α is any formula involving k variables, we have

$$F_{ab}\left[\alpha(x_1, x_2, \ldots, x_k)\right] = \alpha\left[F_{ab}(x_1), F_{ab}(x_2), \ldots, F_{ab}(x_k)\right]$$

for every substitution of elements $x_1, x_2, \ldots, x_k$ of K for the variables of α.

Proof. First we establish the equations

(a) $F_{ab}(x \times y) = F_{ab}(x) \times F_{ab}(y)$,

(b) $F_{ab}(-x) = -F_{ab}(x)$,

(c) $F_{ab}(*x) = *F_{ab}(x)$.

These are easily proved by considering the possible cases. We shall prove (c) here as an example. We shall assume that $x \neq 0$ since otherwise the proof is trivial. If $a \leq x$ and $b \leq x$ or if $a \nleq x$ and $b \nleq x$, then $F_{ab}(*x) = F_{ab}(1) = 1 = *F_{ab}(x)$. If $a \leq x$ and $b \nleq x$, then $F_{ab}(*x) = F_{ab}(1) = 1 = *\left[(x \quad -a) + b\right] = *F_{ab}(x)$. The proof for $a \nleq x$ and $b \leq x$ is similar.

The lemma now follows by induction on the length of α.

<u>Lemma 8</u>. If $\mathcal{M} = \langle K, D, \chi, -, * \rangle$ is a finite Henle matrix and α is a formula containing k variables which is satisfied by $\mathcal{M}$, then $\alpha(x_1, x_2, \dots, x_k) = 1$ for every substitution of elements $x_1, x_2, \dots, x_k$ of K for the variables of α.

<u>Proof</u>. Suppose, if possible, there exists a formula α which is satisfied by $\mathcal{M}$ and such that $\alpha(x_1, x_2, \dots, x_k) = z \neq 1$ for some substitution of elements of K for the variables of α. Since α is satisfied by $\mathcal{M}$, $z \in D$; and, therefore, $d \leq z$ where $d = \prod_{x \in D} x$. Let a be an atom of d and b an atom of $-z$. Then by Lemma 7

$$\alpha\left[F_{ab}(x_1), F_{ab}(x_2), \dots, F_{ab}(x_k)\right] = F_{ab}\left[\alpha(x_1, x_2, \dots, x_k)\right]$$
$$= F_{ab}(z) = (z \chi -a) + b.$$

Since $d \nleq (z \chi -a) + b$, $(z \chi -a) + b \notin D$ and $\mathcal{M}$ does not satisfy α, which is a contradiction.

<u>Definition 4</u>. Two matrices $\mathcal{M}$ and $\mathcal{M}'$ are said to be <u>equivalent</u> if and only if they satisfy the same formulas.

Theorem 3. If $\mathfrak{M} = \langle K, D, X, -, * \rangle$ is a finite Henle matrix, then $\mathfrak{M}$ is equivalent to $\mathfrak{M}' = \langle K, \{1\}, X, -, * \rangle$.

Proof. If α is a formula satisfied by $\mathfrak{M}$, then by Lemma 8 $\alpha = 1$ for every substitution of elements of K for the variables of α; and, therefore, α is satisfied by $\mathfrak{M}'$. If, on the other hand, α is satisfied by $\mathfrak{M}'$, then $\alpha = 1$ for every substitution of elements of K for the variables of α. Hence α is satisfied by $\mathfrak{M}$, since 1 is always a designated element.

Lemma 9. Let $\mathcal{H}_1, \mathcal{H}_2, \ldots, \mathcal{H}_n, \ldots$ be a sequence of Henle matrices such that, for each n, $\mathcal{H}_n$ has n atoms and only one designated element, then

(i) If α is satisfied by $\mathcal{H}_k$, then α is satisfied by $\mathcal{H}_{n-i}$ for $i = 1, 2, \ldots, k-1$.

Proof. This lemma follows from the fact that $\mathcal{H}_k$, for each k, contains every preceding matrix as a sub-matrix.

Lemma 10. If $\mathfrak{M} = \langle K, D, X, -, * \rangle$ is a finite normal S5-matrix, then $\mathfrak{M}$ is equivalent to a finite Henle matrix with one designated element.

Proof. By Theorem 2, $\mathfrak{M}$ is isomorphic, and thus equivalent, to the direct product of a finite sequence of finite Henle matrices, $\mathcal{H}_1, \ldots, \mathcal{H}_k$. By Theorem 3, each of the matrices of this sequence is equivalent to a finite Henle matrix with one designated element; and, thus, the direct product is equivalent to the direct product of a finite sequence of finite Henle matrices with one designated element, $\mathcal{H}'_1, \ldots, \mathcal{H}'_k$. Since this sequence is finite, there must be one member of the sequence which contains a maximum number of elements. By Lemma 9 we see that $\mathfrak{M}$ is equivalent to this matrix.

The principal results of the paper now follow easily from the theorems and lemmas established.

Theorem 4. If S is a consistent quasi-normal extension of S5, and S is not identical with S5, then there exists a finite characteristic matrix for S.

Proof. Let $\mathcal{H}_1, \mathcal{H}_2, \ldots, \mathcal{H}_n, \ldots$ be the sequence of Henle matrices defined as in the proof of Lemma 9, and let G be the set of integers such that $n \in G$ if and only if $\mathcal{H}_n$ is an S-matrix. First suppose G contains infinitely many integers; then, by Lemma 9, G must contain all the integers. Hence S is identical with S5.

Next, suppose that G contains only a finite number of integers, then by Lemma 9, G must contain only integers less than or equal to k for some k. By construction, $\mathcal{H}_k$ is an S-matrix. Now suppose that α is not provable in S; then by Theorem 1, α is not satisfied by some finite, normal S-matrix; and, hence, by Lemma 10, α is not satisfied by $\mathcal{H}_i$ for some $i \in G$. By Lemma 9, α is not satisfied by $\mathcal{H}_k$; and, thus, $\mathcal{H}_k$ is a finite characteristic matrix for S.

Theorem 5. Every quasi-normal extension of S5 is a normal extension of S5.

Proof. McKinsey and Tarski have proved that S5 is a normal extension of itself,[22] and consequently we will deal here only with extensions which are distinct from S5. If S is a quasi-normal extension of S5 and is distinct from S5, then by Theorem 4 there exists a finite characteristic matrix, $\mathfrak{M}$, for S which is a finite Henle matrix with one designated element. Thus if α is provable in S, then $\alpha = 1$ for every substitution of elements of $\mathfrak{M}$ for the variables of α. Since $-*-1 = -*0 = -0 = 1$, the formula $\sim\Diamond\sim\alpha = 1$ for every substitution of elements of $\mathfrak{M}$ for the variables of α. Thus $\sim\Diamond\sim\alpha$ is satisfied by $\mathfrak{M}$ and must be provable in S since $\mathfrak{M}$ is characteristic. Hence, by definition, S is a normal extension of S5.

Theorem 6. If in the infinite matrix $\mathfrak{M} = \langle K, \{1\}, \times, -, * \rangle$:

(i) K is a boolean algebra with respect to $-$ and $\times$,

(ii) $*x = 1$ for $x \neq 0$,

(iii) $*0 = 0$,

then $\mathfrak{M}$ is a characteristic matrix for S5.

Proof. It is easily verified that $\mathfrak{M}$ is a normal S5-matrix. Moreover, $\mathfrak{M}$ contains every matrix of the sequence $\mathcal{H}_1, \mathcal{H}_2, \ldots, \mathcal{H}_n, \ldots$ as a sub-matrix. Since every formula not provable in S5 fails for one of the matrices of this sequence, it must also fail for $\mathfrak{M}$. This completes the proof of the theorem.

In proving that there is no finite characteristic matrix for S5, Dugundji made use of a sequence of formulas,

$$p_1 \equiv p_2,$$
$$(p_1 \equiv p_2) \vee (p_1 \equiv p_3) \vee (p_2 \equiv p_3).$$ [23]

It is interesting that these formulas can be used to axiomatize all possible normal extensions of S5.

To simplify the notation in this discussion, we let P_n represent the formula of Dugundji which contains exactly n distinct variables, and $\mathcal{H}_k$ the Henle matrix with one designated element which contains 2^k elements. It is clear that P_n fails for a Henle matrix with one designated element when and only when it is possible to substitute distinct elements of the matrix for the n variables of P_n. Thus P_n is satisfied by $\mathcal{H}_k$ when 2^k is less than n, and is not satisfied by $\mathcal{H}_k$ when 2^k is equal to or greater than n. By Theorem 4, the characteristic matrix for the extension of S5 formed by adding as an axiom the formula P_n will be $\mathcal{H}_k$ where k is the greatest integer such that 2^k is less than n. Also, the extension having $\mathcal{H}_{2^n}$ as a characteristic matrix can be formed by adding to the axioms of S5 any one of the formulas P_m, where $2^n < m \leq 2^{n+1}$. We note here that, while S5 has an infinite number of normal extensions, every normal extension of S5 distinct from S5 has only a finite number of normal extensions.

By a complete extension of a system S, we denote an extension S' such that every proper extension of S' is inconsistent. Using a very general result of Tarski, McKinsey showed[24] that S5 has only one complete extension. This result can be obtained here without the use of this theorem. It is clear that any extension of S5 can be further extended to the sentential calculus by adding P_3 as an axiom. Hence by definition S5 has the sentential calculus as its only complete extension.

FOOTNOTES

1) This paper was prepared under the direction of Dr. J. C. C. McKinsey as a Master's thesis at Oklahoma Agricultural and Mechanical College and was submitted April 30, 1950.

2) See Dugundji [2]. (The numbers in square brackets refer to the bibliography at the end of the paper.)

3) See Lewis and Langford [4], pp. 122-178, and pp. 492-502.

4) See McKinsey and Tarski [7], p. 7.

5) For a discussion of the rules of substitution and detachment, see Lewis and Langford [4], pp. 125-126.

6) See footnote (5).

7) For definitions see Tarski [9], pp. 103-109.

8) See McKinsey [5], p. 118.

9) See Huntington [3], p. 292, or McKinsey [5], p. 118.

10) See McKinsey [5], p. 122.

11) See McKinsey [5], p. 123.

12) See McKinsey and Tarski [7], p. 9, Theorem 3.9 .

13) For a definition of the <u>degree</u> of a formula see Parry [8], p. 144, footnote (11).

14) See McKinsey [5], p. 119.

15) For proof that $*0 = 0$ in normal S4-matrices, see McKinsey [5], p. 127.

16) For a proof that $*x + *y = *(x+y)$ in normal S2-matrices, see McKinsey [5], p. 119.

17) This follows easily from McKinsey's proof that the class D of designated elements is an additive ideal. See McKinsey [5], p. 120.

18) See McKinsey [5], p. 119.

19) See Lewis and Langford [4], Appendix II, p. 492, footnote (1).

20) A similar result was obtained by Bergmann. See Bergmann [1].

21) See footnote (17).

22) See McKinsey and Tarski [7], p. 5, Theorem 2.1 .

23) See Dugundji [2].

24) See McKinsey [6].

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